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On Schur complements of sign regular matrices of order k

Rong Huang *,[1], Jianzhou Liu [2]

School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, Hunan, China

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The issue regarding Schur complements of sign regular matrices is rather subtle. It is known that the class of totally nonnegative matrices is not closed under arbitrary Schur complementation. In this paper, we demonstrate how Schur complements of sign regular matrices of order k are sign regular of a certain order. In particular, some results for totally nonnegative and totally nonpositive matrices are provided as our corollaries.

1. Introduction

Given $k \in \langle n \rangle = \{1, 2, \ldots, n\}$. Define

$$Q_{k,n} = \{\omega = (\omega_1, \ldots, \omega_k) | 1 \leqslant \omega_1 < \cdots < \omega_k \leqslant n\}.$$

Let $A \in \mathbb{R}^{n\times n}$ and $\alpha, \beta \in Q_{k,n}$. Then $A[\alpha|\beta]$ is by definition the submatrix with rows indexed by α and columns indexed by β, and $A(\alpha|\beta)$ is the submatrix with rows indexed by $\langle n\rangle\backslash\alpha$ and columns

* Corresponding author.
E-mail addresses: rongh98@yahoo.cn (R. Huang), liujz@xtu.edu.cn (J. Liu).

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indexed by $\langle n\rangle\backslash\beta$. When $\alpha=\beta$, $A[\alpha|\alpha]$ and $A(\alpha|\alpha)$ are abbreviated to $A[\alpha]$ and $A(\alpha)$, respectively. For convenience, write $A(\alpha)=A(r)$ if $\alpha=(r)$.

For a given k, a vector $\eta=(\epsilon_1,\ldots,\epsilon_k)$ is called a signature sequence if $k\leqslant n$ and $|\epsilon_i|=1$ ($i=1,\ldots,k$). If $\epsilon_r\det A[\alpha|\beta]\geqslant(>)0$ for all $\alpha,\beta\in Q_{r,n}$ ($r=1,\ldots,k$), then A is called *(strictly) sign regular of order k with signature* η. When $k=n$, A is simply called *(strictly) sign regular*. In particular, if $\epsilon_i=1$ for $i=1,\ldots,n$, then A is *totally nonnegative (totally positive)*; and if $\epsilon_i=-1$ for $i=1,\ldots,n$, then A is *totally nonpositive (totally negative)*.

Sign regular matrices appear in many fields such as approximation theory, combinatorics, statistics and economics. One of the interesting properties of these matrices is that they can be characterized by some variation-diminishing properties which are useful for shape-preserving representations in computer-aided geometric design [7]. Recently, some properties and characterizations on sign regular matrices have been studied in [2,3,5]. Our motivation of this paper is to give some interesting results for Schur complements of sign regular matrices.

If $A[\alpha|\beta]$ is nonsingular, then the Schur complement of $A[\alpha|\beta]$ in A is given by

$$A/[\alpha|\beta]=A/A[\alpha|\beta]=A[\alpha'|\beta']-A[\alpha'|\beta](A[\alpha|\beta])^{-1}A[\alpha|\beta']$$

where $\alpha'=\langle n\rangle\backslash\alpha$ and $\beta'=\langle n\rangle\backslash\beta$. In particular, if $\alpha=(\alpha_i),\beta=(\beta_i)\in Q_{k,n}$, then we write $A/[\alpha|\beta]=A/\{\alpha_1,\ldots,\alpha_k|\beta_1,\ldots,\beta_k\}$. When $\alpha=\beta$, we will use A/α for $A/[\alpha|\beta]$. Schur complements have been well-studied for various classes of matrices such as positive definite matrices, M-matrices and inverse M-matrices. It is well known that the classes of positive definite matrices, M-matrices and inverse M-matrices are closed under arbitrary Schur complementation. But the issue regarding Schur complements of sign regular matrices is rather subtle. For a totally nonnegative (or totally nonpositive) matrix A, it is shown in [1,4–6] that A/α is totally nonnegative only if $\langle n\rangle\backslash\alpha$ is an index set consisting of consecutive integers. Otherwise, A/α is not totally nonnegative as the following example shows: let a totally nonnegative matrix

$$A=\begin{pmatrix}2&1&1&1\\3&2&3&3\\1&1&4&4\\1&1&5&6\end{pmatrix},$$

then

$$A/\{2\}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{5}{2}&\frac{5}{2}\\-\frac{1}{2}&\frac{7}{2}&\frac{9}{2}\end{pmatrix}\text{ is not totally nonnegative.}$$

However, if we set $S_n^{(k)}=\operatorname{diag}(x_t)$ is an n-by-n diagonal matrix where

$$x_t=\begin{cases}-1&\text{if } t\leqslant k,\\1&\text{otherwise,}\end{cases}$$

then it is surprising to get that $S_3^{(1)}(A/\{2\})S_3^{(1)}$ is totally nonnegative. In addition,

$$A/\{1|4\}=\begin{pmatrix}-3&-1&0\\-7&-3&0\\-11&-5&-1\end{pmatrix},$$

but it is interesting to check that $(A/\{1|4\})S_3^{(3)}$ is totally nonnegative.

The aim of this paper is to provide a general result for Schur complements of sign regular matrices of order k. Our main result is the following theorem.

Theorem 1. *Suppose $A\in\mathbb{R}^{n\times n}$ is sign regular of order k ($k\geqslant 2$) with signature $\eta=(\epsilon_1,\ldots,\epsilon_k)$. Set $\alpha=(\alpha_i),\beta=(\beta_i)\in Q_{t,n}$ with $t<k$. If $A[\alpha|\beta]$ is nonsingular, then*

$$\left(S_{n-t}^{(\alpha_t-t)}\cdots S_{n-t}^{(\alpha_1-1)}\right)(A/[\alpha|\beta])\left(S_{n-t}^{(\beta_1-1)}\cdots S_{n-t}^{(\beta_t-t)}\right)$$

is sign regular of order $k - t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t}, \ldots, \frac{\epsilon_k}{\epsilon_t}\right)$. In particular, if $\langle n\rangle\backslash\alpha$ is an index set consisting of consecutive integers, then A/α is sign regular of order $k - t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t}, \ldots, \frac{\epsilon_k}{\epsilon_t}\right)$.

Next we provide an example to illustrate Theorem 1.

Example. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 5 & 5 & 5 & 4 & 1 \\ 8 & 8 & 7 & 5 & 1 \\ 13 & 12 & 9 & 6 & 1 \end{pmatrix}.$$

Then it is not difficult to check that A is sign regular with signature $(1, -1, -1, 1, 1)$. By a simple calculation, we obtain

$$A/\{1, 4|2, 4\} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ 1 & -1 & 3 \end{pmatrix}$$

which, clearly, is not sign regular. However, it does hold that

$$S_3^{(2)}S_3^{(0)}(A/\{1, 4|2, 4\})S_3^{(1)}S_3^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{5}{3} \\ 1 & 1 & 3 \end{pmatrix}$$

is sign regular with signature $(1, -1, -1)$.

The following corollary is directly derived from Theorem 1.

Corollary 2. *Suppose $A \in \mathbb{R}^{n\times n}$ is totally nonnegative or totally nonpositive. Set $\alpha = (\alpha_i)$, $\beta = (\beta_i) \in Q_{t,n}$ with $t < n$. If $A[\alpha|\beta]$ is nonsingular, then*

$$\left(S_{n-t}^{(\alpha_t-t)} \cdots S_{n-t}^{(\alpha_1-1)}\right)(A/[\alpha|\beta])\left(S_{n-t}^{(\beta_1-1)} \cdots S_{n-t}^{(\beta_t-t)}\right)$$

is totally nonnegative. In particular, if $\langle n\rangle\backslash\alpha$ is an index set consisting of consecutive integers, then A/α is totally nonnegative.

2. Proof of the main result

Lemma 3. *Suppose $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is sign regular of order k $(k \geqslant 2)$ with signature $\eta = (\epsilon_1, \ldots, \epsilon_k)$. If $a_{t_1,t_2} \neq 0$, then*

$$S_{n-1}^{(t_1-1)}(A/\{t_1|t_2\})S_{n-1}^{(t_2-1)}$$

is sign regular of order $k - 1$ with signature $\left(\frac{\epsilon_2}{\epsilon_1}, \ldots, \frac{\epsilon_k}{\epsilon_1}\right)$.

Proof. To prove the conclusion, we need to show

$$\frac{\epsilon_{l+1}}{\epsilon_1} \cdot \det\left(S_{n-1}^{(t_1-1)}(A/\{t_1|t_2\})S_{n-1}^{(t_2-1)}\right)[\alpha|\beta] \geqslant 0 \tag{1}$$

for all $\alpha = (\alpha_i)$, $\beta = (\beta_i) \in Q_{l,n-1}$ with any $l \in \langle k-1\rangle = \{1, 2, \ldots, k-1\}$. Set $\alpha' = \tilde{\alpha} \cup \{t_1\}$ and $\beta' = \tilde{\beta} \cup \{t_2\}$, where $\tilde{\alpha} = (\tilde{\alpha}_i)$ and $\tilde{\beta} = (\tilde{\beta}_i)$ satisfy the following

$$\tilde{\alpha}_i = \begin{cases} \alpha_i & \text{if } t_1 > \alpha_i, \\ \alpha_i + 1 & \text{if } t_1 \leqslant \alpha_i, \end{cases} \quad \text{and} \quad \tilde{\beta}_i = \begin{cases} \beta_i & \text{if } t_1 > \beta_i, \\ \beta_i + 1 & \text{if } t_1 \leqslant \beta_i. \end{cases}$$

Now consider the $(l+1)\times(l+1)$ submatrix $B=(b_{ij})=A[\alpha'|\beta']$. Assume that a_{t_1,t_2} is in the position (r,s) of B, i.e, $a_{t_1,t_2}=b_{rs}$. This means that

$$\alpha_1<\cdots<\alpha_{r-1}<t_1\leqslant\alpha_r<\cdots<\alpha_l \tag{2}$$

and

$$\beta_1<\cdots<\beta_{s-1}<t_2\leqslant\beta_s<\cdots<\beta_l. \tag{3}$$

So $(A/\{t_1|t_2\})[\alpha|\beta]=B/\{r|s\}$. Thus it is not difficult to get that

$$\det A[\alpha'|\beta']=\det B=(-1)^{r+s}b_{rs}\det(B/\{r|s\})=(-1)^{r+s}a_{t_1,t_2}\det(A/\{t_1|t_2\})[\alpha|\beta].$$

By (2) and (3), we have that $S_{n-1}^{(t_1-1)}[\alpha]=S_l^{(r-1)}$ and $S_{n-1}^{(t_2-1)}[\beta]=S_l^{(s-1)}$. Hence,

$$\begin{aligned}\det\left(S_{n-1}^{(t_1-1)}(A/\{t_1|t_2\})S_{n-1}^{(t_2-1)}\right)[\alpha|\beta]&=\det\left(S_{n-1}^{(t_1-1)}[\alpha]\right)\det(A/\{t_1|t_2\})[\alpha|\beta]\det(S_{n-1}^{(t_2-1)}[\beta]\big)\\&=(-1)^{r+s}\det(A/\{t_1|t_2\})[\alpha|\beta]\\&=\frac{1}{a_{t_1,t_2}}\det A[\alpha'|\beta'],\end{aligned}$$

from which it is easy to get that (1) holds. $\square$

Lemma 4. *Let A be an n-by-n matrix and D be an n-by-n nonsingular diagonal matrix. Then*

$$(AD)/\{r|s\}=(A/\{r|s\})D(s),\ (DA)/\{r|s\}=D(r)(A/\{r|s\}).$$

Further,

$$(D_1AD_2)/\{r|s\}=D_1(r)(A/\{r|s\})D_2(s)$$

for nonsingular diagonal matrices D_1 and D_2.

Proof. Let $A=(a_{ij})$ and $D=\operatorname{diag}(x_i)$ with all $x_i\neq 0$. We first consider the case that $r=1$ and $s=n$ if $a_{1n}\neq 0$. Then

$$(AD)/\{1|n\}=\begin{pmatrix}x_1a_{21}&\cdots&x_{n-1}a_{2,n-1}\\ \vdots&\cdots&\vdots\\ x_1a_{n1}&\cdots&x_{n-1}a_{n,n-1}\end{pmatrix}-\frac{1}{x_na_{1n}}\begin{pmatrix}x_na_{2n}\\ \vdots\\ x_na_{nn}\end{pmatrix}(x_1a_{11},\ldots,x_{n-1}a_{1,n-1})$$

from which it follows that $(AD)/\{1|n\}=(A/\{1|n\})D(n)$. Thus, using a similar way, we easily get that $(AD)/\{r|s\}=(A/\{r|s\})D(s)$ for any $a_{rs}\neq 0$. Similarly, $(DA)/\{r|s\}=D(r)(A/\{r|s\})$. Clearly,

$$(D_1AD_2)/\{r|s\}=D_1(r)(AD_2)/\{r|s\}=D_1(r)(A/\{r|s\})D_2(s)$$

for nonsingular diagonal matrices D_1 and D_2. $\square$

To prove our result, we require the important result Theorem 1.3 of [1] as follows.

Lemma 5 [1]. *Let A be an n-by-n matrix, and suppose that $A[\alpha|\beta]$ is nonsingular for $\alpha,\beta\in Q_{t,n}$. If $\omega,\tau\in Q_{k,n}$, $\omega\subset\langle n\rangle\backslash\alpha$ and $\tau\subset\langle n\rangle\backslash\beta$, then*

$$(A/[\alpha|\beta])/[\omega|\tau]=A/[\alpha\cup\omega|\beta\cup\tau].$$

For a nonsingular sign regular matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$ of order $k(k\geqslant 2)$ with signature $(\epsilon_1,\ldots,\epsilon_k)$, it is necessary to be pointed out that if $\epsilon_2=1$, then $a_{11}\neq 0$; and if $\epsilon_2=-1$, then $a_{1n}\neq 0$. We are now ready to prove Theorem 1.

Proof of Theorem 1. To prove the result, we use the induction method on the order n of A. It is easy to check that the cases $n=1,2$ are true. Now assume that the assertion holds for the orders less than n.

Since $A \in \mathbb{R}^{n\times n}$ is sign regular of order k and $t < k$, we have that $A[\alpha|\beta] \in \mathbb{R}^{t\times t}$ is sign regular, from which it follows that $a_{\alpha_1,\beta_1} \neq 0$ or $a_{\alpha_t,\beta_1} \neq 0$. Thus we consider two cases as follows.

- If $a_{\alpha_1,\beta_1} \neq 0$, by Lemma 3, then

$$B = S_{n-1}^{(\alpha_1-1)}(A/\{\alpha_1|\beta_1\})S_{n-1}^{(\beta_1-1)}$$

is sign regular of order $k-1$ with signature $\left(\frac{\epsilon_2}{\epsilon_1},\ldots,\frac{\epsilon_k}{\epsilon_1}\right)$. Set $\tilde{\alpha} = (\alpha_2-1,\ldots,\alpha_t-1)$ and $\tilde{\beta} = (\beta_2-1,\ldots,\beta_t-1)$. Then it is not difficult to get that $B[\tilde{\alpha}|\tilde{\beta}] \in \mathbb{R}^{(t-1)\times(t-1)}$ is nonsingular since $A[\alpha|\beta]$ is nonsingular. Consider the fact that $\alpha_1-1 < \alpha_2-1 < \cdots < \alpha_t-1$ and $\beta_1-1 < \beta_2-1 < \cdots < \beta_t-1$. Thus, it holds by Lemmas 4 and 5 that

$$\begin{aligned} B/[\tilde{\alpha}|\tilde{\beta}] &= S_{n-1}^{(\alpha_1-1)}(\tilde{\alpha})(A/\{\alpha_1\ldots,\alpha_t|\beta_1,\ldots,\beta_t\})S_{n-1}^{(\beta_1-1)}(\tilde{\beta}) \\ &= S_{n-t}^{(\alpha_1-1)}(A/[\alpha|\beta])S_{n-t}^{(\beta_1-1)}. \end{aligned}$$

Now applying the induction assumption to $B \in \mathbb{R}^{(n-1)\times(n-1)}$, we get that

$$\begin{aligned} &\left(S_{(n-1)-(t-1)}^{((\alpha_t-1)-(t-1))}\cdots S_{(n-1)-(t-1)}^{((\alpha_2-1)-1)}\right)(B/[\tilde{\alpha}|\tilde{\beta}])\left(S_{(n-1)-(t-1)}^{((\beta_2-1)-1)}\cdots S_{(n-1)-(t-1)}^{((\beta_t-1)-(t-1))}\right) \\ &= \left(S_{n-t}^{(\alpha_t-t)}\cdots S_{n-t}^{(\alpha_2-2)}\right)(B/[\tilde{\alpha}|\tilde{\beta}])\left(S_{n-t}^{(\beta_2-2)}\cdots S_{n-t}^{(\beta_t-t)}\right) \\ &= \left(S_{n-t}^{(\alpha_t-t)}\cdots S_{n-t}^{(\alpha_1-1)}\right)(A/[\alpha|\beta])\left(S_{n-t}^{(\beta_1-1)}\cdots S_{n-t}^{(\beta_t-t)}\right) \end{aligned}$$

is sign regular of order $k-t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t},\ldots,\frac{\epsilon_k}{\epsilon_t}\right)$.

- If $a_{\alpha_t,\beta_1} \neq 0$, by Lemma 3, then

$$B = S_{n-1}^{(\alpha_t-1)}(A/\{\alpha_t|\beta_1\})S_{n-1}^{(\beta_1-1)}$$

is sign regular of order $k-1$ with signature $\left(\frac{\epsilon_2}{\epsilon_1},\ldots,\frac{\epsilon_k}{\epsilon_1}\right)$. Set $\tilde{\alpha} = (\alpha_1,\ldots,\alpha_{t-1})$ and $\tilde{\beta} = (\beta_2-1,\ldots,\beta_t-1)$. Then it is not difficult to get that $B[\tilde{\alpha}|\tilde{\beta}] \in \mathbb{R}^{(t-1)\times(t-1)}$ is nonsingular. Consider the fact that $\alpha_1 < \cdots < \alpha_{t-1} \leqslant \alpha_t - 1$ and $\beta_1-1 < \beta_2-1 < \cdots < \beta_t-1$. Thus, it holds by Lemmas 4 and 5 that

$$\begin{aligned} B/[\tilde{\alpha}|\tilde{\beta}] &= S_{n-1}^{(\alpha_t-1)}(\tilde{\alpha})(A/\{\alpha_1,\ldots,\alpha_t|\beta_1,\ldots,\beta_t\})S_{n-1}^{(\beta_1-1)}(\tilde{\beta}) \\ &= S_{n-t}^{(\alpha_t-t)}(A/[\alpha|\beta])S_{n-t}^{(\beta_1-1)}. \end{aligned}$$

Now applying the induction assumption to $B \in \mathbb{R}^{(n-1)\times(n-1)}$, we get that

$$\begin{aligned} &\left(S_{(n-1)-(t-1)}^{(\alpha_{t-1}-(t-1))}\cdots S_{(n-1)-(t-1)}^{(\alpha_1-1)}\right)(B/[\tilde{\alpha}|\tilde{\beta}])\left(S_{(n-1)-(t-1)}^{((\beta_2-1)-1)}\cdots S_{(n-1)-(t-1)}^{((\beta_t-1)-(t-1))}\right) \\ &= \left(S_{n-t}^{(\alpha_{t-1}-(t-1))}\cdots S_{n-t}^{(\alpha_1-1)}\right)(B/[\tilde{\alpha}|\tilde{\beta}])\left(S_{n-t}^{(\beta_2-2)}\cdots S_{n-t}^{(\beta_t-t)}\right) \\ &= \left(S_{n-t}^{(\alpha_{t-1}-(t-1))}\cdots S_{n-t}^{(\alpha_1-1)}\right)\left(S_{n-t}^{(\alpha_t-t)}(A/[\alpha|\beta])S_{n-t}^{(\beta_1-1)}\right)\left(S_{n-t}^{(\beta_2-2)}\cdots S_{n-t}^{(\beta_t-t)}\right) \\ &= \left(S_{n-t}^{(\alpha_t-t)}\cdots S_{n-t}^{(\alpha_1-1)}\right)(A/[\alpha|\beta])\left(S_{n-t}^{(\beta_1-1)}\cdots S_{n-t}^{(\beta_t-t)}\right) \end{aligned}$$

is sign regular of order $k-t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t},\ldots,\frac{\epsilon_k}{\epsilon_t}\right)$.

Therefore, it is proved that the conclusion is true. In particular, if $\langle n\rangle\backslash\alpha \in Q_{n-t,n}$ is an index set consisting of consecutive integers, then we can assume that $\langle n\rangle\backslash\alpha = (r+1, r+2, \ldots, n-t+r)$ for some integer r, which implies that $\alpha = (1, \ldots, r) \cup (n-t+r+1, \ldots, n)$. Thus it is not difficult to get that

$$\left(S_{n-t}^{(\alpha_t - t)} \cdots S_{n-t}^{(\alpha_1 - 1)}\right)(A/\alpha)\left(S_{n-t}^{(\alpha_1 - 1)} \cdots S_{n-t}^{(\alpha_t - t)}\right) = A/\alpha.$$

Hence, we conclude that if $\langle n\rangle\backslash\alpha$ is an index set consisting of consecutive integers, then A/α is sign regular of order $k-t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t}, \ldots, \frac{\epsilon_k}{\epsilon_t}\right)$. $\square$

Remark. Given a matrix $A = (a_{ij})$, we denote by $|A|$ the matrix of absolute values of the entries of A. We easily get from Theorem 1 that if $A \in \mathbb{R}^{n\times n}$ is sign regular of order k $(k \geqslant 2)$ with signature $\eta = (\epsilon_1, \ldots, \epsilon_k)$, then $\frac{\epsilon_{t+1}}{\epsilon_t} \cdot |A/[\alpha|\beta]|$ is sign regular of order $k-t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t}, \ldots, \frac{\epsilon_k}{\epsilon_t}\right)$ for $\alpha, \beta \in Q_{t,n}$ with $t < k$. In particular, if A is totally nonnegative or totally nonpositive, then $|A/[\alpha|\beta]|$ is totally nonnegative for $\alpha, \beta \in Q_{t,n}$ with $t < n$.

According to the proof of Theorem 1, we immediately have the following results.

Theorem 6. *Suppose $A \in \mathbb{R}^{n\times n}$ is strictly sign regular of order $k(k \geqslant 2)$ with signature $\eta = (\epsilon_1, \ldots, \epsilon_k)$. Set $\alpha = (\alpha_i)$, $\beta = (\beta_i) \in Q_{t,n}$ with $t < k$. If $A[\alpha|\beta]$ is nonsingular, then*

$$\left(S_{n-t}^{(\alpha_t - t)} \cdots S_{n-t}^{(\alpha_1 - 1)}\right)(A/[\alpha|\beta])\left(S_{n-t}^{(\beta_1 - 1)} \cdots S_{n-t}^{(\beta_t - t)}\right)$$

is strictly sign regular of order $k-t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t}, \ldots, \frac{\epsilon_k}{\epsilon_t}\right)$. In particular, if $\langle n\rangle\backslash\alpha$ is an index set consisting of consecutive integers, then A/α is strictly sign regular of order $k-t$ with signature $\left(\frac{\epsilon_{t+1}}{\epsilon_t}, \ldots, \frac{\epsilon_k}{\epsilon_t}\right)$.

Corollary 7. *Suppose $A \in \mathbb{R}^{n\times n}$ is totally positive or totally negative. Set $\alpha = (\alpha_i)$, $\beta = (\beta_i) \in Q_{t,n}$ with $t < n$. If $A[\alpha|\beta]$ is nonsingular, then*

$$\left(S_{n-t}^{(\alpha_t - t)} \cdots S_{n-t}^{(\alpha_1 - 1)}\right)(A/[\alpha|\beta])\left(S_{n-t}^{(\beta_1 - 1)} \cdots S_{n-t}^{(\beta_t - t)}\right)$$

is totally positive. In particular, if $\langle n\rangle\backslash\alpha$ is an index set consisting of consecutive integers, then A/α is totally positive.

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